

ANALOGUES OF LEHMER'S PROBLEM IN POSITIVE CHARACTERISTIC

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ABSTRACT. Let C be a smooth projective irreducible curve defined over a finite field \mathbb{F}_q and $K = \mathbb{F}_q(C)$. We show that every non-torsion element $\alpha \in \overline{K}$ of degree d over K of a Drinfeld A -module ϕ defined over K has canonical height $\hat{h}_\phi(\alpha)$ at least $1/d$. Similarly, if E/K is a non-constant elliptic curve defined over a function field $K = l(C)$ of a curve C defined over an algebraically closed field l of characteristic 0 or $p > 3$, we show that every point of infinite order $P \in E(\overline{K})$ of degree d over K has canonical height $\hat{h}_E(P)$ at least c/d , where c depends only on the degree of the j -map associated to E/K .

1. INTRODUCTION

Let α be an algebraic number of degree d over \mathbb{Q} and suppose it is not a root of unity. Let $h : \overline{\mathbb{Q}} \rightarrow \mathbb{R}$ be the absolute logarithmic height. Lehmer's conjecture consists in asking for an absolute real constant $c > 0$ such that $h(\alpha) \geq \frac{c}{d}$.

Although this question remains open, analogues of this conjecture have been considered in other contexts. Let E be an elliptic curve defined over a number field K , j_E its j -invariant, \overline{K} the algebraic closure of K , $P \in E(\overline{K})$ a point of infinite order and $\hat{h}_E : E(\overline{K}) \rightarrow \mathbb{R}$ its canonical height. Let $K(P)$ be the field generated over K by the coordinates of P , $d = [K(P) : K]$ and $D = [K : \mathbb{Q}]$. In [5, Corollary 0.2] it is shown that if j_E is non-integral, then there exists $c > 0$ depending on E/K such that $\hat{h}_E(P) \geq \frac{c}{d^2(\log d)^2}$. Let $h = \max\{1, h(j_E)\}$. This result was improved in [1, Corollary 1.4], where it was proved that there exist absolute effective computable real constants $c_5, c_6 > 0$ such that $\hat{h}_E(P) \geq c_5 h(dD)^{-3} \left(1 + \frac{\log(dD)}{h}\right)^{-2}$, if j_E is integral, and $\hat{h}_E(P) \geq c_6 D^{-3} d^{-15/8} h^{-2} \left(1 + \frac{\log(dD)}{h}\right)^{-2}$, otherwise. In section 3 we prove an analogue of this result for non-constant elliptic curves over function fields over algebraically closed fields of characteristic 0 or $p > 3$.

Let C be a smooth irreducible projective curve defined over a finite field \mathbb{F}_q of q elements and $K = \mathbb{F}_q(C)$. The direct translation of Lehmer's conjecture to K is trivial, because the requirement that α is not a root of unity is equivalent to $\alpha \in K - \mathbb{F}_q$, thus there is a discrete valuation $v : K \rightarrow \mathbb{Z} \cup \{\infty\}$ of K such that $v(\alpha) < 0$ and therefore $h(\alpha) \geq \frac{1}{d}$.

Another instance of the Lehmer problem is to consider the canonical height $\hat{h}_\phi : \overline{K} \rightarrow \mathbb{R}$ of a Drinfeld A -module $\phi : A \rightarrow K\{\tau\}$ of rank r defined over a $K = \mathbb{F}_q(C)$. We also have a notion of torsion elements in this context and we ask for a constant c depending on ϕ such that for every non-torsion element $\alpha \in \overline{K}$

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with $[K(\alpha) : K] = d$ we have $\hat{h}_\phi(\alpha) \geq \frac{c}{d}$. We prove this in section 2 starting with the case where $K = \mathbb{F}_q(T)$ is the rational function field over \mathbb{F}_q and $A = \mathbb{F}_q[T]$. The general result is then deduced from this case. Analogues of this type of result were proved in [3].

2. DRINFELD MODULES

Let $A = \mathbb{F}_q[T]$ be the polynomial ring in one variable over the finite field \mathbb{F}_q of q elements, $k = \mathbb{F}_q(T)$ its field of fractions and $\phi : A \rightarrow \text{End}_k(\mathbb{G}_a) \cong k\{\tau\}$ a Drinfeld A -module of rank r defined over k with respect to the inclusion $A \subset k$. Denote $\phi_T = T + a_1\tau + \dots + a_r\tau^r$.

Let \bar{k} be the algebraic closure of k and $h : \bar{k} \rightarrow \mathbb{R}$ the absolute logarithmic Weil height. The global height of the Drinfeld module ϕ at $\alpha \in \bar{k}$ is defined by (cf. [2, §2])

$$\hat{h}_\phi(\alpha) = \lim_{n \rightarrow \infty} \frac{h(\phi_{T^n}(\alpha))}{q^{nr}}.$$

Analogously to the case of elliptic curves this global height decomposes in a sum of local heights which are defined as follows. Let $L = k(\alpha)$ and M_L the set of places of L normalized so that they correspond to discrete valuations $v : L \rightarrow \mathbb{Z} \cup \{\infty\}$. Let d_v be the degree of v and $d = [L : k]$. The local height of α at v with respect to ϕ is defined by (cf. [9, §4])

$$\hat{h}_{\phi,v}(\alpha) = -\frac{d_v}{d} \lim_{n \rightarrow \infty} \frac{\min\{0, v(\phi_{T^n}(\alpha))\}}{q^{nr}}.$$

It follows from the above definitions that

$$(2.1) \quad \hat{h}_\phi(\alpha) = \sum_{v \in M_L} \hat{h}_{\phi,v}(\alpha).$$

An element $\alpha \in \bar{k}$ is called a torsion element of ϕ if there exists $f \in A - \{0\}$ such that $\phi_f(\alpha) = 0$.

Theorem 2.1. *Let $\alpha \in \bar{k}$ be a non-torsion element of ϕ and $d = [k(\alpha) : k]$. Then*

$$\hat{h}_\phi(\alpha) \geq \frac{1}{d}.$$

Proof. Let $S \subset M_L$ be the set consisting of the poles of $T = a_0, a_1, \dots, a_r$ and the zeros of a_r . Suppose there exists $v \notin S$ such that $v(\alpha) < 0$. Then, for every $0 \leq i < d$,

$$q^r v(\alpha) + v(a_r) = q^r v(\alpha) < q^i v(\alpha) \leq q^i v(\alpha) + v(a_i),$$

hence $v(\phi_T(\alpha)) = q^r v(\alpha)$. By induction, for every $n \geq 1$, we also have $v(\phi_{T^n}(\alpha)) = q^{nr} v(\alpha)$, thus $\hat{h}_{\phi,v}(\alpha) = -\frac{d_v}{d} v(\alpha) \geq \frac{1}{d}$.

Assume now that all the poles of α lie in S . Let $v \in S$ be a pole of α . Let

$$M_{\phi,v} = \min_{0 \leq i < r} \frac{v(a_i) - v(a_r)}{q^r - q^i}.$$

Suppose $v(\alpha) < M_{\phi,v}$ and $v(a_r) \leq 0$. The first inequality implies $v(\phi_T(\alpha)) = v(a_r) + q^r v(\alpha)$. The two inequalities imply

$$q^r (q^r - q^i) v(\alpha) < (q^r - q^i) v(\alpha) < v(a_i) - v(a_r) \leq v(a_i) - v(a_r) - (q^r - q^i) v(a_r),$$

for every $0 \leq i < r$, i.e.,

$$q^{2r} v(\alpha) + (q^r + 1) v(a_r) < q^{r+i} v(\alpha) + q^i v(a_r) + v(a_i),$$

i.e.,

$$v(\phi_{T^2}(\alpha)) = q^{2r}v(\alpha) + (q^r + 1)v(a_r) = q^{2r}v(\alpha) + \frac{q^{2r} - 1}{q^r - 1}v(a_r).$$

Suppose we have proved that for every integer $1 \leq m < n$ we have

$$v(\phi_{T^m}(\alpha)) = q^{mr}v(\alpha) + \frac{q^{mr} - 1}{q^r - 1}v(a_r).$$

Then

$$\begin{aligned} & q^r(q^r - q^i)v(\phi_{T^{n-2}}(\alpha)) \\ &= q^r(q^r - q^i)(q^{(n-2)r}v(\alpha) + (q^{(n-3)r} + \dots + q^r + 1)v(a_r)) \\ &\leq q^{(n-1)r}(q^r - q^i)v(\alpha) < (q^r - q^i)v(\alpha) < v(a_i) - v(a_r) \\ &\leq v(a_i) - v(a_r) - (q^r - q^i)v(a_r). \end{aligned}$$

Thus,

$$\begin{aligned} & q^{nr}v(\alpha) + (q^{(n-1)r} + \dots + q^r + 1)v(a_r) \\ &< q^{(n-1)r+i}v(\alpha) + (q^{(n-2)r+i} + \dots + q^{r+i} + q^i)v(a_r) + v(a_i), \end{aligned}$$

i.e.,

$$v(\phi_{T^n}(\alpha)) = q^{nr}v(\alpha) + (q^{(n-1)r} + \dots + q^r + 1)v(a_r) = q^{nr}v(\alpha) + \frac{q^{nr} - 1}{q^r - 1}v(a_r).$$

Hence,

$$\hat{h}_{\phi,v}(\alpha) = -\frac{d_v}{d} \left(v(\alpha) + \frac{1}{q^r - 1}v(a_r) \right) \geq \frac{1}{d}.$$

Suppose now that $v(\alpha) < M_{\phi,v}$, but $v(a_r) > 0$. Let ξ be a sufficiently negative power of a local parameter at v so that $v(\xi^{q^r-1}a_r) \leq 0$. The Drinfeld module $\psi = \xi^{-1}\phi\xi$ is isomorphic to ϕ and by [9, Proposition 2] $\hat{h}_{\phi,v} = \hat{h}_{\psi,v}$. Note that $\psi_T = T + \xi^{q-1}a_1\tau + \dots + \xi^{q^r-1}a_r\tau^r$. Then for every $0 \leq i < r$ we have

$$(q^r - q^i)v(\alpha) < v(a_i) - v(a_r) < v(a_i) - v(a_r) - v(\xi)(q^r - q^i) = v(\xi^{q^i-1}a_i) - v(\xi^{q^r-1}a_r),$$

in particular, $v(\alpha) < M_{\psi,v}$. By the argument of the last paragraph we conclude that $\hat{h}_{\phi,v}(\alpha) = \hat{h}_{\psi,v}(\alpha) \geq \frac{1}{d}$.

If $v(\alpha) \geq M_{\phi,v}$, let ξ be a sufficiently positive power of a local parameter at v such that

$$M_{\psi,v} = \min_{0 \leq i < r} \frac{v(a_i\xi^{1-q^i}) - v(a_r\xi^{1-q^r})}{q^r - q^i} = \min_{0 \leq i < r} \left(\frac{v(a_i) - v(a_r)}{q^r - q^i} + v(\xi) \right) > v(\alpha).$$

Once again we take the Drinfeld module $\psi = \xi^{-1}\phi\xi$ which is isomorphic to ϕ . By the two last cases and [9, Proposition 2] we conclude that $\hat{h}_{v,\phi}(\alpha) = \hat{h}_{v,\psi}(\alpha) \geq \frac{1}{d}$.

By the non-negativity of the local canonical heights we conclude that $\hat{h}_{\phi}(\alpha) \geq \frac{1}{d}$. \square

2.1. The general case. The Lehmer problem for Drinfeld modules can be formulated in a more general set-up and its proof is reduced to that of Theorem 2.1. Let C be a smooth projective irreducible curve defined over a finite field \mathbb{F}_q of q elements. Let ∞ be a fixed place of $K = \mathbb{F}_q(C)$, A the ring of functions in K which are regular everywhere except at ∞ , $v_\infty : K \rightarrow \mathbb{Z} \cup \{\infty\}$ the normalized discrete valuation associated to ∞ and d_∞ the degree of ∞ . For any $a \in A$, let $\deg(a) = -d_\infty v_\infty(a)$. The field K is an A -module with respect to the inclusion $A \subset K$. A Drinfeld A -module of rank r defined over K is a ring homomorphism

$\phi : A \rightarrow \text{End}_K(\mathbb{G}_a) \cong K\{\tau\}$ such that for every $a \in A$, $\deg(\phi_a) = q^{r \deg(a)}$ and the constant term of ϕ_a is a itself.

Let $a \in A - \mathbb{F}_q$. The global height of $\alpha \in \overline{K}$ is defined as (cf. [2, §2])

$$\hat{h}_\phi(\alpha) = \lim_{n \rightarrow \infty} \frac{h(\phi_{a^n}(\alpha))}{\deg(\phi_{a^n})}.$$

Let $L = K(\alpha)$ and $d = [L : K]$. For every discrete valuation $v : L \rightarrow \mathbb{Z} \cup \{\infty\}$ of degree d_v the local height is defined as (cf. [9, §4])

$$\hat{h}_{\phi,v}(\alpha) = \lim_{n \rightarrow \infty} -\frac{d_v \min\{0, v(\phi_{a^n}(\alpha))\}}{d \deg(\phi_{a^n})}.$$

As observed in [9, Proposition 3] these heights are independent of the choice of $a \in A - \mathbb{F}_q$.

The Dedekind domain A is a finitely generated \mathbb{F}_q -algebra. Let \mathcal{A} the the set of generators of A as an \mathbb{F}_q -algebra, $T \in \mathcal{A}$, $\deg(T) = d_T$ and $\phi_T = T + a_1(T)\tau + \dots + a_{rd_T}(T)\tau^{rd_T}$. Let $\alpha \in \overline{K}$ be a non-torsion element for ϕ with $[K(\alpha) : K] = d$. Replacing the a_i 's in the proof of Theorem 2.1 by the $a_i(T)$'s the proof of Theorem 2.1 shows that

$$(2.2) \quad \hat{h}_\phi(\alpha) \geq \frac{1}{d}.$$

3. ELLIPTIC CURVES

Let C be a smooth irreducible projective curve defined over an algebraically closed field l of characteristic 0 or $p > 3$, let $K = l(C)$ be its function field and \overline{K} its algebraic closure. Let E/K be a non-constant semistable elliptic curve defined over K , $\varphi_E : \mathcal{E} \rightarrow C$ its minimal semi-stable regular model, $j_E : C \rightarrow \mathbb{P}^1$ the j -map induced by φ_E and $\hat{h}_E : E(\overline{K}) \rightarrow \mathbb{R}$ its canonical height.

Let $P \in E(\overline{K})$ and $L = K(P)$ the field generated by K and the coordinates of P . Let $d = [L : K]$, M_L the set of places v of L which are normalized so that $v : L \rightarrow \mathbb{Z} \cup \{\infty\}$ is the corresponding discrete valuation. Let L_v be the completion of L with respect to v and $\lambda_v : E(K_v) \rightarrow \mathbb{R}$ its local Néron function [11, Chapter VI]. Let $w = v|_K$, $e(v|w)$ the ramification index of v over w , $w' = e(v|w)w : K \rightarrow \mathbb{Z} \cup \{\infty\}$ the normalization of w , K_w the completion of K with respect to w and $n(v|w) = [L_v : K_w]$. Then

$$(3.1) \quad \hat{h}_E(P) = \frac{1}{d} \sum_{v \in M_L} n(v|w) \lambda_v(P),$$

[11, VI, Theorem 2.1].

Let $\mathfrak{D}_{E/K}$ be the minimal discriminant of E/K and $d_{E/K} = \deg(\mathfrak{D}_{E/K})$. Since E/K is semi-stable, it follows from [10, Chapter VII, Proposition 5.1] that $w'(\mathfrak{D}_{E/K}) = -w'(j_E)$ for every pole w' of j_E , thus $\deg(j_E) = d_{E/K}$. For every $v \in M_L$, let $v^+ = \max\{v, 0\}$.

Lemma 3.1. [5, Proposition 1.3] *Let $A, N \geq 1$ be integers, $Q_0, \dots, Q_{6AN} \in E(L_v)$ distinct points. Then there exists $P_0, \dots, P_N \in \{Q_0, \dots, Q_{6AN}\}$ such that for each $i \neq l$,*

$$\lambda_v(P_i - P_l) \geq \frac{1 - A^{-1}}{12} v^+(j_E^{-1}).$$

Proposition 3.2. $\#\{Q \in E(L) \mid \hat{h}_E(Q) < \frac{d_{E/K}}{96d}\} \leq 24$.

Proof. Denote $\mathcal{S} = \{Q \in E(L) \mid \hat{h}_E(Q) < \frac{d_{E/K}}{96d}\}$ and suppose $\#\mathcal{S} > 24$. Let $A = 2$ and $N + 1 = \lfloor \frac{\#\mathcal{S}}{12} \rfloor > 1$ the integral part of $\frac{\#\mathcal{S}}{12}$, then $1 < N + 1 \leq \frac{\#\mathcal{S}}{12}$. So we can choose $12N + 1$ distinct points Q_0, \dots, Q_{12N+1} in \mathcal{S} . By Lemma 3.1 there exist $P_0, \dots, P_N \in \{Q_0, \dots, Q_{12N+1}\}$ such that $\lambda_v(P_i - P_l) \geq \frac{1}{24}v^+(j_{\mathcal{E}}^{-1})$ for $i \neq l$. It follows from the triangle inequality that

$$(3.2) \quad H = \max_{Q \in \mathcal{S}} \hat{h}_E(Q) \geq \max_{1 \leq i \leq N} \hat{h}(P_i) \geq \frac{1}{4N(N+1)} \sum_{i \neq l} \hat{h}_E(P_i - P_l).$$

Hence, by (3.1) and (3.2),

$$(3.3) \quad \begin{aligned} H &\geq \frac{1}{4N(N+1)d} \sum_{i \neq l} \sum_{v \in M_L} n(v|w) \lambda_v(P_i - P_l) \geq \frac{1}{96d} \sum_{v \in M_L} n(v|w) v^+(j_{\mathcal{E}}^{-1}) \\ &= \frac{1}{96d} \sum_{w' \in M_K} \sum_{v|w} \frac{n(v|w)}{e(v|w)} w'^+(j_{\mathcal{E}}^{-1}) \geq \frac{1}{96d} \sum_{w' \in M_K} w'^+(j_{\mathcal{E}}^{-1}) = \frac{d_{E/K}}{96d}. \end{aligned}$$

□

Remark 3.3. We used the fact that l is algebraically closed just to ensure that the poles of $j_{\mathcal{E}}$ have all degree 1.

As a consequence of Proposition 3.2 we obtain a theorem which simultaneously deals with the Lehmer and the Lang problems for elliptic curves over function fields. Recall that the Lang problem is to find a constant $c > 0$ depending on E/K such that for every non-torsion point $P \in E(K)$ we have $\hat{h}_E(P) \geq cd_{E/K}$.

Theorem 3.4. *Let $P \in E(\overline{K})$ be a non-torsion point of E/K and $d = [K(P) : K]$. Then there exists an absolute real constant $c > 0$ such that $\hat{h}_E(P) \geq c \frac{d_{E/K}}{d}$.*

Proof. Suppose $\hat{h}_E(P) < \frac{d_{E/K}}{60000d}$. Then for every $1 \leq n \leq 25$, $\hat{h}_E(nP) = n^2 \hat{h}_E(P) < \frac{d_{E/K}}{96d}$, which contradicts Proposition 3.2. So we take $c = \frac{1}{60000}$. □

Remark 3.5. The constant for the Lehmer problem is $\frac{d_{E/K}}{60000}$ so it depends only on $\deg(j_{\mathcal{E}}) = d_{E/K}$, in the semi-stable case.

Remark 3.6. In [6, Theorem 0.2] Hindry and Silverman proved Lang's conjecture for function fields over algebraically closed fields of characteristic 0. In the case where $d_{E/K} \geq 24(g-1)$, where g denotes the genus of K , they obtained an absolute constant c . However, our constant is greater than theirs, thus improving the result. In the case where $d_{E/K} < 24(g-1)$, their constant depends exponentially on g , whereas ours is absolute and improves the constant part of their bound. Nevertheless, we have just proved Lang's conjecture in the case of semi-stable elliptic curves. Inspired on [6, Theorem 0.2] we had previously proved Lang's conjecture for semi-stable elliptic curves over function fields of positive characteristic [7, Theorem 5] using [5, Proposition 1.2]. First, the bounds we obtained there do not have absolute constants, they depended not only on g but also on the inseparable degree of $j_{\mathcal{E}}$. Furthermore, the present bound improves their constant parts. The reason for obtaining an absolute constant is that [5, Proposition 1.3] gives a lower bound which depends only on the choice of a positive integer A , however the lower bound of [5, Proposition 1.2] depends on the number $N + 1$ of points P_0, \dots, P_N chosen in $E(L_v)$ (cf. [7, proof of Proposition 3]).

Another consequence of Proposition 3.2 is a bound for the order of the torsion group $E(K)_{\text{tor}}$.

Corollary 3.7. $\#E(K)_{\text{tor}} \leq 24$.

Remark 3.8. Previous bounds for the torsion of elliptic curves over function fields in characteristic 0 were obtained in [6, Theorem 7.2] and in the case of characteristic p , Goldfeld and Szpiro treated the case where C is defined over a finite field [4, Theorem 13], but the result extends to algebraically closed fields of characteristic $p > 3$ and we also obtained a bound (cf. [7, Theorem 7]) using [7, Proposition 3]. In the case of characteristic 0, the upper bound depended on $d_{E/K}$. Using Szpiro's theorem on the minimal discriminant of elliptic curves over function fields [13, Théorème 1], i.e., $d_{E/K} \leq 6p^e(2g - 2 + f_{E/K})$, where p^e is the inseparable degree of j_E and $f_{E/K}$ is the degree of the conductor divisor of E/K , it follows an upper bound whose constant part is worse than the bound of Corollary 3.7. The bounds in characteristic p (in the semi-stable case) were $\sigma_{E/K}^2$, respectively $2\sigma_{E/K}^2$, where $\sigma_{E/K} = \frac{d_{E/K}}{f_{E/K}}$. If $d_{E/K} \geq 24p^e(g - 1)$, then (using again [13, Théorème 1]) $\sigma_{E/K} \leq 12p^e$ and otherwise $\sigma_{E/K} \leq d_{E/K} < 24p^e(g - 1)$. Not only is the bound of Corollary 3.7 absolute, but also it is better than the estimates for $\sigma_{E/K}^2$.

3.1. Integral points. Theorem 3.4 and Corollary 3.7 imply as in [6, §8] an upper bound for the number of integral points of an S -minimal Weierstrass equation of E/K .

Let S be a finite set of places of K and $R_S \subset K$ the ring of S -integers. For every $a \in K$ let $h_K(a) = [K : l(a)]$. A Weierstrass equation $y^2 = x^3 + Bx + C$ with discriminant Δ is called S -minimal if $h_K(\Delta)$ is minimal subject to $f(x) \in R_S[x]$. Let $\delta = \min\{\hat{h}_E(P) \mid P \in E(K) - (E(K)_{\text{tor}} \cap E(R_S))\}$ and $\epsilon = \max\{\hat{h}_E(P) \mid P \in E(R_S)\}$. In [12, Lemma 1.2 (a)] it is shown that $\#E(R_S) \leq \#E(K)_{\text{tor}}(K)(1 + 2\sqrt{\frac{\epsilon}{\delta}})^{r_E}$, where r_E denotes the rank of $E(K)$. It follows from [7, Remark 14] that

$$(3.4) \quad \epsilon \leq p^e(12g + 4\#S + 5d_{E/K}).$$

Theorem 3.9. *Let $y^2 = x^3 + Bx + C$ be an S -minimal Weierstrass equation for E/K . If $d_{E/K} \geq 24p^e(g - 1)$, then $\#E(R_S) \leq 24(2299\sqrt{p^e\#S})^{r_E}$, otherwise $\#E(R_S) \leq 24(2021\sqrt{gp^e\#S})^{r_E}$.*

Proof. By Theorem 3.4, $\delta \geq \frac{d_{E/K}}{60000}$. If $d_{E/K} \geq 24p^e(g - 1)$, then $g \leq \frac{d_{E/K}}{24p^e} + 1$. Thus, since $\#S \geq 1$,

$$(3.5) \quad \begin{aligned} \frac{\epsilon}{\delta} &\leq 60000 \frac{p^e}{d_{E/K}} (12g + 4\#S + 5d_{E/K}) \\ &\leq 60000p^e \left(12 \left(\frac{1}{24p^e} + 1 \right) + 9\#S \right) \\ &\leq 1320000p^e\#S. \end{aligned}$$

The first statement follows from (3.5) and Corollary 3.7.

Suppose now that $d_{E/K} < 24p^e(g - 1)$. In this case, since $\#S \geq 1$ and $g \geq 2$, we have

$$(3.6) \quad \begin{aligned} \frac{\epsilon}{\delta} &\leq 60000 \frac{p^e}{d_{E/K}} (12g + 4\#S + 5d_{E/K}) \leq 60000p^e(12g + 9\#S) \\ &\leq 1020000p^eg\#S. \end{aligned}$$

The second statement follows from (3.6) and Corollary 3.7. \square

Remark 3.10. The bound of Theorem 3.9 improves the bounds of [6, Theorem 8.1] in the case of characteristic 0 when $d_{E/K} \geq 24(g-1)$. When $d < 24(g-1)$ we also have an improvement of the constant part (which does not depend on the rank r_E of $E(K)$) if $g \geq 3$. Note that in this latter case, the constant part of their bound depends on g , whereas ours does not. In both cases the bound of Theorem 3.9 improves that of [7, Theorem 15].

3.2. Lehmer problem : the general case. If we no longer suppose that E/K is a semi-stable elliptic curve, then $\deg(j_{\mathcal{E}}) < d_{E/K}$ (cf. [10, Chapter VII, Proposition 5.1]). In this case, instead of Proposition 3.2, we need to bound the cardinality of a smaller set

$$(3.7) \quad \# \left\{ Q \in E(L) ; \hat{h}_E(Q) < \frac{\deg(j_{\mathcal{E}})}{96d} \right\} \leq 24.$$

As a consequence, Theorem 3.4 is replaced by: for every non-torsion point P of E of degree d over K we have $\hat{h}_E(P) \geq \frac{c'}{d}$, where $c' = \frac{\deg(j_{\mathcal{E}})}{60000}$. We cannot obtain Lang's conjecture as in Theorem 3.4, because Lemma 3.1 involves $v^+(j_{\mathcal{E}}^{-1})$, hence the proof of Proposition 3.2 only gives $\deg(j_{\mathcal{E}})$ and not $d_{E/K}$.

3.3. Integral points : the general case. The bound of (3.7) also implies that $\#E(K)_{\text{tor}} \leq 24$. Note that in the general case

$$f_{E/K} < 2\#\{\text{poles of } j_{\mathcal{E}}\} \leq 2\deg_s(j_{\mathcal{E}}),$$

where $\deg_s(j_{\mathcal{E}})$ denotes the separable degree of $j_{\mathcal{E}}$.

Theorem 3.11. *Let $y^2 = x^3 + Bx + C$ be an S -minimal Weierstrass equation for E/K . If $d_{E/K} \geq 24p^e(g-1)$, then $\#E(R_S) \leq 24(13788\sqrt{gp^e\#S})^{r_E}$, otherwise $\#E(R_S) \leq 24(12121g\sqrt{p^e\#S})^{r_E}$.*

Proof. If $d_{E/K} \geq 24p^e(g-1)$, then

$$(3.8) \quad \begin{aligned} \frac{\epsilon}{\delta} &\leq 60000 \frac{p^e}{\deg(j_{\mathcal{E}})} (12g + 4\#S + 5d_{E/K}) \\ &\leq 60000 \frac{p^e}{\deg(j_{\mathcal{E}})} \left(\frac{d_{E/K}}{2p^e} + 12 + 9d_{E/K}\#S \right) \\ &\leq 1320000 \frac{p^e}{\deg(j_{\mathcal{E}})} d_{E/K} \#S. \end{aligned}$$

Szpiro's discriminant theorem [13, Théorème 1] was first proved in the case of semi-stable elliptic curves. However, this result was extended by Pesenti and Szpiro to any elliptic curve [8, Théorème 0.1]. It follows from [8, Théorème 0.1], (3.8) and $f_{E/K} < 2\deg_s(j_{\mathcal{E}})$ that

$$(3.9) \quad \begin{aligned} \frac{\epsilon}{\delta} &\leq 7920000 \frac{p^{2e}}{\deg(j_{\mathcal{E}})} (2g - 2 + f_{E/K}) \#S \\ &\leq 23760000 \frac{p^{2e}}{\deg(j_{\mathcal{E}})} g f_{E/K} \#S \leq 47520000 p^e g \#S. \end{aligned}$$

The result now follows from (3.9) and $\#E(K)_{\text{tor}} \leq 24$.

Suppose now that $d_{E/K} < 24p^e(g-1)$, then

$$\begin{aligned}
 (3.10) \quad \frac{\epsilon}{\delta} &\leq 60000 \frac{p^e}{\deg(j_{\mathcal{E}})} (12g + 4\#S + 5d_{E/K}) \leq 1020000 \frac{p^e}{\deg(j_{\mathcal{E}})} g d_{E/K} \#S \\
 &\leq 6120000 \frac{p^{2e}}{\deg(j_{\mathcal{E}})} g(2g - 2 + f_{E/K}) \#S \leq 18360000 \frac{p^{2e}}{\deg(j_{\mathcal{E}})} g^2 f_{E/K} \#S \\
 &\leq 36720000 p^e g^2 \#S.
 \end{aligned}$$

The result now follows from (3.10) and $\#E(K)_{\text{tor}} \leq 24$. \square

Remark 3.12. Observe that the bounds of Theorem 3.11 are a worse than those of Theorem 3.9.

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